$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\sum_{k=0}^{\infty}\frac{\left(-\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}\pi x_i x_j+2\sum_{i=1}^{n}\sqrt{\sum_{j_0=0}^{i}\binom{i}{j_0}\frac{\pi\cos((2i-j_0)\pi)}{i\sum_{j_1=0}^{i-j_0}\sum_{j_2=0}^{j_1}\cdots\sum_{j_{11}=0}^{j_{10}}\binom{i-j_0}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{10}}{j_{11}}}}\,x_i\right)^k}{k!}\,dx_n\cdots dx_1=12$$

発想

多変数 Gauss 積分 (多変数正規分布の確率密度関数などに使用される) の公式 (証明省略):

$$\int\exp\left(-\frac{1}{2}\mathbf{x}^{\mathrm{T}}A\mathbf{x}+\mathbf{b}^{\mathrm{T}}\mathbf{x}\right)d\mathbf{x}=\sqrt{\frac{(2\pi)^n}{\det A}}\exp\left(\frac{1}{2}\mathbf{b}^{\mathrm{T}}A^{-1}\mathbf{b}\right)$$

について,

$$A=\begin{pmatrix}2\pi & & 0\\ & \ddots & \\ 0 & & 2\pi\end{pmatrix}=2\pi E_n$$

$$\mathbf{b}=\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix},\quad b_i=2\sqrt{\frac{\pi}{i}\left(\frac{11}{1+11}\right)^i}$$

とし, 左辺を成分ごとに書き, Maclaurin 展開により exp を $\sum$ に直し, 二項定理により 12^{j_0} を $\sum$ と二項係数を利用して表し, $(-1)^{2i-j_0}$ を三角関数 (cos) を利用して書き, $n\to\infty$ という極限をとれば上式の左辺となる.

多変数 Gauss 積分の右辺については,

$$A^{-1}=\frac{1}{2\pi}E_n,\quad \det A=(2\pi)^n$$

より, 以下のように計算できる:

$$\begin{aligned}\sqrt{\frac{(2\pi)^n}{\det A}}\exp\left(\frac{1}{2}\mathbf{b}^{\mathrm{T}}A^{-1}\mathbf{b}\right) &= \sqrt{\frac{(2\pi)^n}{(2\pi)^n}}\exp\left(\frac{1}{2}\cdot\frac{1}{2\pi}\mathbf{b}^{\mathrm{T}}\mathbf{b}\right)\\ &= \exp\left(\frac{1}{4\pi}\sum_{i=1}^{n}{b_i}^2\right)\\ &= \exp\left(\frac{1}{4\pi}\sum_{i=1}^{n}4\cdot\frac{\pi}{i}\left(\frac{11}{1+11}\right)^i\right)\\ &= \exp\left(\sum_{i=1}^{n}\frac{1}{i}\left(\frac{11}{1+11}\right)^i\right)\end{aligned}$$

ここで $\log(1+x)$ の Maclaurin 展開が,

$$\log(1+x)=\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}x^i\quad(|x|<1)$$

であり, $x=-\dfrac{y}{1+y}$ とする変換によって,

$$\log(1+y)=\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{y}{1+y}\right)^i\quad\left(-\frac{1}{2}<y\right)$$

とできる ($\dfrac{y}{1+y}=1-\dfrac{1}{1+y}$ とする. 計算詳細省略). よって,

$$\lim_{n\to\infty}\exp\left(\sum_{i=1}^{n}\frac{1}{i}\left(\frac{11}{1+11}\right)^i\right)=\exp\left(\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i}\left(\frac{11}{1+11}\right)^i\right)=\exp(\log(1+11))=12$$

(変数の独立性, 収束半径や極限と関数の入れ替え可能性などの議論は省略).

$$
\begin{aligned}
12^k &= \sum_{j_1=0}^{k} \binom{k}{j_1} 11^{j_1} 1^{k-j_1} \\
&= \sum_{j_1=0}^{k} \left(\binom{k}{j_1} \sum_{j_2=0}^{j_1} \binom{j_1}{j_2} 10^{j_2} 1^{j_1-j_2} \right) \\
&\vdots \\
&= \sum_{j_1=0}^{k} \left(\binom{k}{j_1} \sum_{j_2=0}^{j_1} \left(\binom{j_1}{j_2} \sum_{j_3=0}^{j_2} \cdots \sum_{j_{10}=0}^{j_9} \binom{j_9}{j_{10}} 2^{j_{10}} 1^{j_9-j_{10}} \cdots \right) \right) \\
&= \sum_{j_1=0}^{k} \left(\binom{k}{j_1} \sum_{j_2=0}^{j_1} \left(\binom{j_1}{j_2} \sum_{j_3=0}^{j_2} \cdots \sum_{j_{10}=0}^{j_9} \left(\binom{j_9}{j_{10}} \sum_{j_{11}=0}^{j_{10}} \binom{j_{10}}{j_{11}} 1^{j_{11}} 1^{j_{10}-j_{11}} \right) \cdots \right) \right) \\
&= \sum_{j_1=0}^{k} \left(\binom{k}{j_1} \sum_{j_2=0}^{j_1} \left(\binom{j_1}{j_2} \sum_{j_3=0}^{j_2} \cdots \sum_{j_{10}=0}^{j_9} \left(\binom{j_9}{j_{10}} \sum_{j_{11}=0}^{j_{10}} \binom{j_{10}}{j_{11}} \right) \cdots \right) \right) \\
&= \sum_{j_1=0}^{k} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{11}=0}^{j_{10}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{10}}{j_{11}}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{11}{1+11} \right)^i = \left(1 - \frac{1}{12} \right)^i &= \sum_{j_0=0}^{i} \binom{i}{j_0} \frac{(-1)^{i-j_0}}{12^{i-j_0}} \\
&= \sum_{j_0=0}^{i} \binom{i}{j_0} \frac{(-1)^{i-j_0}}{\sum_{j_1=0}^{i-j_0} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{11}=0}^{j_{10}} \binom{i-j_0}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{10}}{j_{11}}}
\end{aligned}
$$

$$
\begin{aligned}
b_i &= 2 \sqrt{ \frac{\pi(-1)^i}{i} \left(\frac{11}{1+11} \right)^i } \\
&= 2 \sqrt{ \sum_{j_0=0}^{i} \binom{i}{j_0} \frac{\pi(-1)^{2i-j_0}}{i \sum_{j_1=0}^{i-j_0} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{11}=0}^{j_{10}} \binom{i-j_0}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{10}}{j_{11}}} } \\
&= 2 \sqrt{ \sum_{j_0=0}^{i} \binom{i}{j_0} \frac{\pi \cos((2i-j_0)\pi)}{i \sum_{j_1=0}^{i-j_0} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{11}=0}^{j_{10}} \binom{i-j_0}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{10}}{j_{11}}} }
\end{aligned}
$$

$$
\mathbf{b}^T \mathbf{x} = 2 \sum_{i=1}^{n} \sqrt{ \sum_{j_0=0}^{i} \binom{i}{j_0} \frac{\pi \cos((2i-j_0)\pi)}{i \sum_{j_1=0}^{i-j_0} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{11}=0}^{j_{10}} \binom{i-j_0}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{10}}{j_{11}}} } x_i
$$

$$
-\frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} = -\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \pi x_i x_j + 2 \sum_{i=1}^{n} \sqrt{ \sum_{j_0=0}^{i} \binom{i}{j_0} \frac{\pi \cos((2i-j_0)\pi)}{i \sum_{j_1=0}^{i-j_0} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{11}=0}^{j_{10}} \binom{i-j_0}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{10}}{j_{11}}} } x_i
$$

$$
\exp(-\frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x}) = \sum_{k=0}^{\infty} \frac{ \left(-\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \pi x_i x_j + 2 \sum_{i=1}^{n} \sqrt{ \sum_{j_0=0}^{i} \binom{i}{j_0} \frac{\pi \cos((2i-j_0)\pi)}{i \sum_{j_1=0}^{i-j_0} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{11}=0}^{j_{10}} \binom{i-j_0}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{10}}{j_{11}}} } x_i \right)^k }{k!}
$$

$$
\begin{aligned}
& \int \exp(-\frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x}) d\mathbf{x} \\
= \; & \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \sum_{k=0}^{\infty} \frac{ \left(-\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \pi x_i x_j + 2 \sum_{i=1}^{n} \sqrt{ \sum_{j_0=0}^{i} \binom{i}{j_0} \frac{\pi \cos((2i-j_0)\pi)}{i \sum_{j_1=0}^{i-j_0} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{11}=0}^{j_{10}} \binom{i-j_0}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{10}}{j_{11}}} } x_i \right)^k }{k!} \; dx_n \cdots dx_1
\end{aligned}
$$

$$
\begin{aligned}
& \lim_{n \to \infty} \int \exp(-\frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x}) d\mathbf{x} \\
= \; & \lim_{n \to \infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \sum_{k=0}^{\infty} \frac{ \left(-\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \pi x_i x_j + 2 \sum_{i=1}^{n} \sqrt{ \sum_{j_0=0}^{i} \binom{i}{j_0} \frac{\pi \cos((2i-j_0)\pi)}{i \sum_{j_1=0}^{i-j_0} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{11}=0}^{j_{10}} \binom{i-j_0}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{10}}{j_{11}}} } x_i \right)^k }{k!} \; dx_n \cdots dx_1 \\
= \; & 12
\end{aligned}
$$